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M5-branes and Wilson surfaces

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Abstract: We investigate the M5-brane description of the Wilson surface operators in six-dimensional (2,0) superconformal field theory from AdS/CFT correspondence. We consider the Wilson surface operators in high-dimensional representation, whose description could be M5-brane string soliton solutions in $AdS_7 \times S^4$ background. We construct such string soliton solutions from the covariant M5-brane equations of motion and discuss their properties. The supersymmetry analysis shows that these solutions are half-BPS. We also discuss the subtle issue on the boundary terms.

Keywords: M-Theory, p-branes, AdS-CFT Correspondence.

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1. Introduction

One of the most important achievements in string theory is AdS/CFT correspondence [1], which states that the quantum gravity in Anti-de-Sitter(AdS) spacetime is dual to the large N limit of the superconformal field theory at AdS boundary. It not only opens a new window to study the quantum gravity from its dual field theory, but also supplies a new tool to study the gauge theory from its gravity dual.

The best studied case in AdS/CFT correspondence is AdS_5/SYM_4, where the boundary field theory is an $\mathcal{N} = 4$ super-Yang-Mills gauge theory. Many works have been devoted to the study of the correspondence in this case. Among them, the study of the supersymmetric Wilson loop operators [2, 3] is one of the most interesting issues. The Wilson loop operators in gauge theory is defined to be the holonomy of the gauge field around a contour. Their expectation values characterize the phase of the theory. From AdS/CFT dictionary, the expectation values could be calculated by considering a fundamental string ending on the boundary of AdS along the path specified by the Wilson loop operator. The area of the string worldsheet bounded by the loop may give the expectation value of the Wilson loop operator, after appropriate regularization. However, this is not the whole story. In the field theory side, it has been found that the calculation of the expectation value of

1/2-BPS circular Wilson loop in SYM_4 could be reduced to a corresponding calculation in a zero-dimensional matrix model [4]. There are two remarkable things. One is that though the expectation value of the straight half-BPS Wilson line is exactly one, the one of the circular loop is more involved, being a function of the 't Hooft coupling. The circular loop is related to the straight line by conformal transformation. The difference of the expectation values of two cases comes from the conformal anomaly of the boundary. The other remarkable point is that the calculation through the matrix model gives the expectation value not only to all orders of 't Hooft coupling $\lambda = g^2_{\rm YM}N$ but also to all orders of $1/N$ [5], which is just the string coupling g_s from the correspondence. This indicates that the calculation in gravity side should go beyond the free string limit.

Inspired by the field theory result, it has been found that the all-genus expectation value of the 1/2 BPS Wilson loop operators is better described by the dynamics of the D3-brane [6] or D5-brane [7], not just by the minimal surface of the string world-sheet [3], especially when the charges of the string are large. Simply speaking, for the Wilson loop in the symmetric representation or the multiply wound Wilson loop, the many coincident fundamental strings could interact themselves and be described in terms of the dynamics of D3-branes with electric flux [6]. This is reminiscent of the giant gravitons [8 – 10]. For the Wilson loops in anti-symmetric representation, the interaction of the strings leads to a better description in terms of the dynamics of the D5-brane with electric flux [7]. One may also understand the above picture from Myers effect [11]: the string worldsheet in the five-form field strength background may blow up in the transverse directions to form dielectric brane [12]. The generalization to the Wilson-'t Hooft operators could be found in [13].

Motivated by the success in the Wilson loop case, we are led to consider its cousin in 6-dimensional field theory. In this case, we have AdS_7/CFT_6 correspondence [14], where the CFT_6 is a six-dimensional superconformal field theory. This duality originate from the the description of M5-brane in 11-dimensional M-theory. The near horizon limit of M5-brane gravity solution in 11-dimensional supergravity is $AdS_7 \times S_4$. And the low energy effective field theory of coincident M5-brane is a $(2,0)$-superconformal field theory [15, 16]. This field theory is quite mysterious [17]: its field content is of a tensor multiplet, including a 2-form $B_{\mu\nu}$, four fermions and five scalars; the field strength of 2-form is (anti)self-dual. The existence of the 2-form gauge field implies that there exist string like excitations in the theory. However, there is no lagrangian formulation of the chiral 2-form, even though the chiral theory is still a local interacting field theory [18]. The theory has been suggested to be described by DLCQ matrix theory [19]. In any sense, it has not been well understood. The AdS/CFT correspondence supplies a new tool to probe this nontrivial six-dimensional field theory. The weak version of the correspondence says that the large N limit of the $(2,0)$ field theory is dual to 11D supergravity on $AdS_7 \times S^4$ [1]. Some properties of the field theory has been studied from its dual gravity, including the correlation functions of the chiral primary operators and the Wilson surface operators [2, 20]. The Wilson surface operators in $(2,0)$ theory could be formally defined as [21]

$$W_0(\Sigma) = \exp i \int_\Sigma B^+, \tag{1.1}$$

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where Σ is a two-dimensional surface. From AdS/CFT correspondence, its expectation value could be calculated from the membrane action as

$$< W_0(\Sigma) >= e^{-S} \tag{1.2}$$

where S is the action of the membrane whose worldvolume boundary is Σ. The action is divergent and needs renormalization. Unlike its cousin the Wilson loop, the surface operator has conformal anomaly since even dimensional submanifold observable is conformally anomalous [22]. For example, the membrane action corresponding to the spherical Wilson surface has both quadratic and logarithmic divergences. The existence of logarithmic divergence indicates that the expectation value of Wilson surface may not be well-defined. Unlike the Wilson loop operator, only abelian Wilson surface operator has been discussed in the literature. The field theory study could be found in [23]: the conformal anomaly of abelian Wilson surface operator was calculated in A_1 field theory. It is very hard to consider the nonabelin Wilson surface in the field theory. In [20], from AdS/CFT correspondence, the Wilson surface operators in the fundamental representation has been studied. It would be interesting to see if we can find the M5-brane description of the Wilson surface in higher dimensional representation, just like the case in the Wilson loop. Especially, we want to consider the 1/2-BPS Wilson surface operators. We will show in this paper that this is feasible.

We must be cautious in talking about the BPS Wilson surface in higher dimensional representation, since we have no idea on how to define it rigorously in the field theory. Formally we can define the Wilson surface in representation R to be

$$W_R(\Sigma) = \mathrm{Tr}_R P[\exp i \int_\Sigma (B^+ + \cdots)], \tag{1.3}$$

where $\cdots$ denotes the possible scalar fields and fermions so that we have a 1/2-BPS Wilson surface operators. Without a lagrangian formulation of the chiral 2-form field theory, it is not clear how to find the half-supersymmetric operators and what the expectation value of $W_R(\sigma)$ really means. It could be better to understand the situation from brane picture. The surface Σ could be taken as the intersection of membrane with the M5-branes. In brane configuration, a Wilson surface operator in the rank k symmetric representation, corresponds to a k-wound membrane ending on Σ,[1] while a Wilson surface operator in the rank m antisymmetric representation corresponds to m membranes ending on M5-branes. These are the two most simplest cases, which we will study in this paper. For more general representation, one may get the brane picture from the lesson in the Wilson loop case [24].

From AdS_7/CFT_6 correspondence, the expecation value of the Wilson surface could be calculated by the regularized volume of the membrane ending on the M5-brane with boundary Σ. For the multi-wrapped Wilson surface, the interaction among membranes may induce a blow-up of the membrane to M5-brane wrapping S^3. The dynamics of M5-brane with self-dual field strength encodes the information of the Wilson surface. Analogue to

[1] Strictly speaking, there could be a small difference between symmetric representation and multi-wound surfaces. In the Wilson loop case, this issue has been addressed in [25].

the Wilson line case, the expectation value of the Wilson surfaces could still be calculated from (1.2) but now the action is the M5-brane action with appropriate boundary terms. However, unlike the Dp-brane, the M5-brane dynamics is much more involved. From the M5-brane point of view, the membranes are the self-dual string soliton. The first step in our investigation is to find the self-dual string soliton solution of M5-brane, whose worldvolume is embedded in $AdS_7 \times S^4$ background. In a curved spacetime, the equations of motion of M5-brane looks forbidding and hard to solve. There is no careful discussion on this issue. Most of the discussion on M5-brane string soliton have been focused on the flat spacetime. We will start from the covariant equations of motion and construct the M5-brane soliton solutions corresponding to the higher-dimensional Wilson surfaces. The similar brane configurations has been discussed in [26] in Pasti-Sorokin-Tonin(PST) formalism. Our results are in agreement with the ones in [26].

This investigation is rewarding. It may help us to understand better the dynamics of the M5-brane, give prediction on the expectation value of multi-wrapped Wilson surface, which has not been worked out in six-dimensional (2,0) theory. It could also shed light on the membrane interaction and the possible Myers effect in M-theory.

The paper is organized as follows. In the next section, we give a brief review of the M5-brane equations of motions. In section 3, we work out M5-brane string soliton solutions, whose worldvolume is embedded into AdS_7 and is of topology $AdS_3 \times S^3$. We consider both the straight Wilson surface and the spherical Wilson surface. These soliton solutions describes the Wilson surface operator in the symmetric representation. In section 4, we study another kind of string soliton solutions with the same topology but with S^3 part in S^4. They describe the straight and spherical Wilson surface operators in the antisymmetric representation. For both kinds of solutions, we discuss their properties, including charges, bulk actions, boundary terms. In section 5, we show that our solutions are half-BPS. This suggests that the corresponding Wilson surface operators are also 1/2-BPS in the field theory. We end with the conclusion and discussions. In appendix, we list various connections used in our calculation.

2. The M5-brane equations of motion and actions

In this section, we give a brief review of the M5-brane covariant equations of motions in curved spacetime. For a review on various aspects of M5-brane, see [27].

The M5-brane covariant equations of motion in eleven dimension was first proposed in [28] in superembedding formalism [29], and was then rederived by requiring κ-symmetry of an open M2-brane ending on the M5-brane [30]. For other derivations from various actions, please see [31, 32]. We only care about the bosonic components of the equations, which include the scalar equation and tensor equation. The scalar equation takes the form

$$G^{mn}\nabla_m \mathcal{E}^{\underline{c}}_{\underline{n}} = \frac{Q}{\sqrt{-g}}\epsilon^{m_1\cdots m_6}\left(\frac{1}{6!}H^{\underline{a}}{}_{m_1\cdots m_6} + \frac{1}{(3!)^2}H^{\underline{a}}{}_{m_1m_2m_3}H_{m_4m_5m_6}\right)P_{\underline{a}}{}^{\underline{c}} \tag{2.1}$$

and the tensor equation is of the form

$$G^{mn}\nabla_m H_{npq} = Q^{-1}(4Y - 2(mY + Ym) + mYm)_{pq}. \tag{2.2}$$

Here our notation is as follows: indices from the beginning(middle) of the alphabet refer to frame(coordinate) indices, and the underlined indices refer to target space ones.

Let us spend some time to explain the quantities in the above equation. There exists a self-dual 3-form field strength h_{mnp} on the M5-brane worldvolume. From it, we can define

$$k_m^{\ n} = h_{mpq}h^{npq}, \tag{2.3}$$

$$Q = 1 - \frac{2}{3}\mathrm{Tr}k^2, \tag{2.4}$$

$$m_p^{\ q} = \delta_p^{\ q} - 2k_p^{\ q}, \tag{2.5}$$

$$H_{mnp} = 4Q^{-1}(1+2k)_m^{\ q}h_{qnp} \tag{2.6}$$

Note that h_{mnp} is self-dual with respect to worldvolume metric but not H_{mnp}. The induced metric is simply

$$g_{mn} = \mathcal{E}_m^{\underline{a}}\mathcal{E}_n^{\underline{b}}\eta_{\underline{ab}} \tag{2.7}$$

where

$$\mathcal{E}_m^{\underline{a}} = \partial_m z^{\underline{m}} E_{\underline{m}}^{\underline{a}}. \tag{2.8}$$

Here $z^{\underline{m}}$ is the target spacetime coordinate, which is a function of worldvolume coordinate ξ through embedding, and $E_{\underline{m}}^{\underline{a}}$ is the component of target space vielbein. From the induced metric, we can define another tensor

$$G^{mn} = \left(1 + \frac{2}{3}k^2\right)g^{mn} - 4k^{mn}. \tag{2.9}$$

We also have

$$P_{\underline{a}}^{\ \underline{c}} = \delta_{\underline{a}}^{\underline{c}} - \mathcal{E}_{\underline{a}}^m \mathcal{E}_m^{\ \underline{c}}. \tag{2.10}$$

Note that in the scalar equation of motion, the covariant derivative $\nabla_m \mathcal{E}_{\underline{n}}^{\underline{c}}$ involves not only the Levi-Civita connection of the M5-brane worldvolume but also the spin connections of the target spacetime geometry. More precisely one has

$$\nabla_m \mathcal{E}_{\underline{n}}^{\underline{c}} = \partial_m \mathcal{E}_{\underline{n}}^{\underline{c}} - \Gamma_{mn}^p \mathcal{E}_{\underline{p}}^{\underline{c}} + \mathcal{E}_m^{\underline{a}}\mathcal{E}_n^{\underline{b}}\omega_{\underline{ab}}^{\underline{c}} \tag{2.11}$$

where Γ_{mn}^p is the Christoffel symbol with respect to the induced worldvolume metric and $\omega_{\underline{ab}}^{\underline{c}}$ is the spin connection of the background spacetime.

Moreover, there is a 4-form field strength $H_{\underline{a}_1\cdots\underline{a}_4}$ and its Hodge dual 7-form field strength $H_{\underline{a}_1\cdots\underline{a}_7}$:

$$\begin{aligned} H_4 &= dC_3 \\ H_7 &= dC_6 + \frac{1}{2}C_3 \wedge H_4 \end{aligned} \tag{2.12}$$

The frame indices on H_4 and H_7 in the above equations (2.1), (2.2) have been converted to worldvolume indices with factors of $\mathcal{E}_m^{\underline{c}}$. From them, we can define

$$Y_{mn} = [4 \star \underline{H} - 2(m \star \underline{H} + \star\underline{H}m) + m \star \underline{H}m]_{mn}, \tag{2.13}$$

where

$$\star\underline{H}^{mn} = \frac{1}{4!\sqrt{-g}}\epsilon^{mnpqrs}\underline{H}_{pqrs} \tag{2.14}$$

The field H_{mnp} is defined by

$$H_3 = dA_2 - \underline{C}_3, \tag{2.15}$$

where A_2 is a 2-form gauge potential and $\underline{C}_3$ is the pull-back of the bulk gauge potential. From its definition, H_3 satisfies the Bianchi identity

$$dH_3 = -\underline{H}_4 \tag{2.16}$$

where $\underline{H}_4$ is the pull-back of the target space 4-form flux. Note that different from the 3-form field h_3 which is self-dual on the worldvolume of M5-brane, H_3 satisfies a nonlinear self-duality condition:

$$*H_{mnp} = Q^{-1}G_m^{\ q}H_{npq}. \tag{2.17}$$

This condition could be rewritten in the following form

$$*H_3 = \frac{\partial\mathcal{K}}{\partial H_3}, \tag{2.18}$$

where

$$\mathcal{K} = 2\sqrt{1 + \frac{1}{12}H^2 + \frac{1}{288}(H^2)^2 - \frac{1}{96}H_{abc}H^{bcd}H_{def}H^{efa}}. \tag{2.19}$$

It would be nice to derive the above equations of motion from an action. However, compared to the Dp-brane action, which is just a Dirac-Born-Infeld(DBI)-type action, M5-brane action is much more subtle since it describe a self-interacting chiral 2-form whose field strength is self-dual. In [33, 34], a manifestly supercovariant and kappa-invariant action has been constructed. It contains an auxiliary scalar, from which the self-duality condition could be derived as an equation of motion. The covariant bosonic action is of a DBI-like form[2]

$$S_c = T_5\int d^6x\left(\sqrt{-\det(g_{mn} + i\tilde{H}_{mn})} - \frac{\sqrt{-g}}{4}\tilde{H}^{mn}H_{mn}\right) - T_5\int Z_6 \tag{2.20}$$

where

$$Z_6 = \underline{C}_6 - \frac{1}{2}\underline{C}_3 \wedge H_3, \tag{2.21}$$

and T_5 is the tension of the M5-brane:

$$T_5 = \frac{1}{(2\pi)^5 l_p^6}. \tag{2.22}$$

Here Z_6 is the Wess-Zumino term, in which $\underline{C}_6$ and $\underline{C}_3$ are the pull-back of the target space gauge potential. In the action, one has

$$\tilde{H}^{mn} = (*H)^{mnp}v_p, \tag{2.23}$$

$$H_{mn} = H_{mnp}v^p, \tag{2.24}$$

[2]For an equivalent formulation with the same philosophy, see [35].

by introducing an auxiliary field b whose normalized derivative is

$$v_p = \frac{\partial_p b}{\sqrt{-g^{mn}\partial_m b \partial_n b}}. \tag{2.25}$$

Note that the vector $\vec{v}$ is timelike, $v_p v^p = -1$ and one has the freedom in choosing v_p. The equation of motion of the auxiliary field b is not independent. It is a consequence of the equation of motion of the 2-form gauge potential, which takes the following form after appropriate gauge fixing:

$$H_{mn} = \mathcal{V}_{mn}, \tag{2.26}$$

where

$$\mathcal{V}_{mn} = -\frac{2}{\sqrt{-g}} \frac{\delta\sqrt{-\det(g_{mn} + i\tilde{H}_{mn})}}{\delta \tilde{H}^{mn}}. \tag{2.27}$$

The relation (2.26) is actually a generalized self-dual condition.

This proposal has some troubles in defining a proper partition function, since the topological class of auxiliary scalar would break some symmetries of M-theory, as pointed out in [18]. The resolution of this problem is to embed the chiral theory into a non-chiral one. In [31], a nonchiral M5-brane action for unconstrained 2-form gauge potential has been constructed. In this action, one has to impose a non-linear self-duality condition to ensure kappa-symmetry. And the equation of motion for 2-form potential is equivalent to the Bianchi identity. The action is given by

$$S = S_{M5} - S_{WZ} = T_5 \int \left(\frac{1}{2} \star \mathcal{K} - Z_6 \right) \tag{2.28}$$

There are two remarkable relation on $\mathcal{K}$, when the nonlinear self-duality condition (2.18) holds:

$$\begin{aligned} \mathcal{K} &= 2K = 2\sqrt{1 + \frac{1}{24} H^{mnp} H_{mnp}}, \\ K &= 2Q^{-1} - 1. \end{aligned} \tag{2.29}$$

It has been proved in [32] that the two different actions (2.20) and (2.28) are equivalent, leading to the same set of equations of motion.

3. M5-brane description of the Wilson surface in the symmetric representation

The $AdS_7 \times S^4$ spacetime could be taken as the near horizon geometry of M5-brane gravity solution. It is also the maximally supersymmetric solution in 11-dimensional supergravity, whose equations of motion take the form,

$$R_{MN} = \frac{1}{2 \times 3!} H_{MPQR} H_N^{\ PQR} - \frac{1}{6 \times 4!} g_{MN} H_{PQRS} H^{PQRS}, \tag{3.1}$$

$$0 = \partial_M \sqrt{-g} H^{MNPQ} + \frac{1}{2 \times (4!)^2} \epsilon^{M_1 \cdots M_8 NPQ} H_{M_1 \cdots M_4} H_{M_5 \cdots M_8} \tag{3.2}$$

where the indices take values from 0 to 10. The metric and background 4-form flux of $AdS_7 \times S^4$ are

$$\begin{aligned} ds^2 &= \frac{R^2}{y^2}(dy^2 - dt^2 + dx^2 + dr^2 + r^2 d\Omega_3^2) + \frac{R^2}{4} d\Omega_4^2 \\ H_4 &= \frac{3R^3}{8} \sin^3\zeta_1 \sin^2\zeta_2 \sin\zeta_3 d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_3 \wedge d\zeta_4 \end{aligned} \tag{3.3}$$

where $d\Omega_3^2$ is the metric of S^3 and $d\Omega_4^2$ is the metric of S^4. The 4-form field strength fills in S^4, in which $\zeta_i, i = 1, 2, 3$ are three angular coordinates in S^4. Note that the radius of AdS_7 is twice the one of S^4. Here since our discussion following will focus on the AdS_7 in this section, we rescale its radius to be R. From the AdS_7/CFT_6 duality, we know that

$$R = (8\pi N)^{\frac{1}{3}} l_p, \tag{3.4}$$

where l_p is the Planck constant in eleven dimensions.

3.1 Straight Wilson surface

The standard description of the Wilson surface in AdS/CFT correspondence is the boundary of a minimal area membrane worldvolume in $AdS_7 \times S^4$ background. For the straight Wilson surface, we can set the membrane worldvolume to be extended in the directions y, t, x and fixed at $r = 0$. From the AdS/CFT dictionary, the expectation value of the Wilson surface operator depends on the volume of the membrane through (1.2). In the straight Wilson surface case, the volume with respect to the induced metric is

$$\begin{aligned} V_2 &= \int \frac{R^3}{y^3} dydtdx \\ &= TX\frac{R^3}{2y_0^2}, \end{aligned} \tag{3.5}$$

where T, X is the lengths of the t, x directions. Here we have introduced a cutoff y_0 to regularize the volume. The action is just

$$S = T_2 V_2 = \frac{N}{\pi} TX \frac{1}{y_0^2}, \tag{3.6}$$

where $T_2 = \frac{1}{(2\pi)^2 l_p^3}$ is the membrane tension. The action is proportional to the area of the surface and is also of quadratic divergence.

To have a M5-brane description of the Wilson surface in the high rank representation, we have to find the appropriate M5-brane solution first. Inspired by the study of the Wilson loop, one may attempt to try a M5-brane with worldvolume $AdS_3 \times S^3$ [36]. The induced membrane worldvolume is an AdS_3 and the blow-up of the background flux gives an S^3. In the case of the straight Wilson surface, let the worldvolume coordinates of M5-brane be $\xi_i, i = 0, \ldots 5$ and the embedding be

$$\xi_0 = t, \quad \xi_1 = x, \quad \xi_2 = r, \quad y = f(r), \tag{3.7}$$

$$\xi_3 = \alpha, \quad \xi_4 = \beta, \quad \xi_5 = \gamma \tag{3.8}$$

where α, β, γ are the angular coordinates of S^3. The induced metric is then

$$\begin{aligned} ds^2_{\rm ind} &= \frac{R^2}{f^2}(-d\xi_0^2 + d\xi_1^2 + (1+f'^2)d\xi_2^2 + r^2 d\Omega_3^2) \\ &= \frac{R^2}{f^2}(-dt^2 + dx^2 + (1+f'^2)dr^2) + \frac{R^2 r^2}{f^2}(d\alpha^2 + \sin^2\alpha d\beta^2 + \sin^2\alpha \sin^2\beta d\gamma^2) \end{aligned} \tag{3.9}$$

where the prime denotes the derivative with respect to r. Without causing confusion, we simply let $t, x, r, \alpha, \beta, \gamma$ be the coordinates of the M5-brane worldvolume.

There is a self-dual 3-form field strength in the M5-brane worldvolume. Let us assume it to be

$$h_3 = \frac{a}{2}(1 + \star_{\rm ind})\sqrt{\det G} d\alpha \wedge d\beta \wedge d\gamma \tag{3.10}$$

where a could be a function of r and $\det G$ is the determinant of the metric of S^3. In our case, we have

$$h_3 = \frac{a}{2}\left(\frac{R}{f}\right)^3 (r^3 \sin^2\alpha \sin\beta d\alpha \wedge d\beta \wedge d\gamma + \sqrt{1+f'^2} dt \wedge dx \wedge dr). \tag{3.11}$$

Then we can calculate the relevant quantities k^{mn}, G^{mn} etc. Here we list the quantities which will be useful to our following discussion:

$$\begin{aligned} k^2 &= k_{mn}k^{mn} = \frac{3}{2}a^4, \\ k_m^{\ n} &= \begin{pmatrix} -\frac{a^2}{2} I_3 & 0 \\ 0 & \frac{a^2}{2} I_3 \end{pmatrix}, \\ G^{tt} &= -G^{xx} = -\left(\frac{f}{R}\right)^2 (1+a^2)^2, \\ G^{rr} &= \left(\frac{f}{R}\right)^2 \frac{(1+a^2)^2}{1+f'^2}, \\ G^{\alpha\alpha} &= \left(\frac{f}{Rr}\right)^2 (1-a^2)^2, \\ G^{\beta\beta} &= \left(\frac{f}{Rr\sin\alpha}\right)^2 (1-a^2)^2 \\ G^{\gamma\gamma} &= \left(\frac{f}{Rr\sin\alpha\sin\beta}\right)^2 (1-a^2)^2, \end{aligned} \tag{3.12}$$

where I_3 is a rank 3 identity matrix, and

$$H_3 = 2a\left(\frac{R}{f}\right)^3 \left(\frac{\sqrt{1+f'^2}}{1+a^2} dt \wedge dx \wedge dr + \frac{r^3}{1-a^2}\sin^2\alpha \sin\beta d\alpha \wedge d\beta \wedge d\gamma\right) \tag{3.13}$$

The non-chiral five-brane action is just

$$S = T_5 \int dt dx dr d\alpha d\beta d\gamma \left(\frac{R}{f}\right)^6 r^3 \sin^2\alpha \sin\beta \left(\sqrt{1+f'^2}\frac{1+a^4}{1-a^4} - 1\right) \tag{3.14}$$

Since there is no pull-back of bulk 4-form field strength on the M5-brane worldvolume, we have $dH_3 = 0$, which gives the constraint

$$\frac{a}{1-a^2}\frac{r^3}{f^3} = \text{constant} \tag{3.15}$$

The equation of motion on the tensor H_{npq}, in this case, is

$$G^{mn}\nabla_m H_{npq} = 0. \tag{3.16}$$

Here ∇_m is the covariant derivative with respect to the induced metric. We list the detailed Levi-Civita connection in appendix. It is straightforward to check that the above equation is satisfied, provided that a is a constant. Then from $dH = 0$, we can determine

$$f(r) = \kappa r, \tag{3.17}$$

where κ is just a constant. This is reminiscent of the solution in the straight Wilson line case [6, 13] in $AdS_5 \times S^5$ and the spiky solution in flat spacetime [37, 38]. Then the induced metric of M5-brane is

$$ds^2 = \frac{R^2}{\kappa^2 r^2}(-dt^2 + dx^2 + (1+\kappa^2)dr^2) + \frac{R^2}{\kappa^2}(d\alpha^2 + \sin^2\alpha d\beta^2 + \sin^2\alpha \sin^2\beta d\gamma^2). \tag{3.18}$$

This indicates that the worldvolume of M5 brane is actually $AdS_3 \times S^3$, with radius $\frac{R\sqrt{1+\kappa^2}}{\kappa}$ in AdS_3 and radius $\frac{R}{\kappa}$ in S^3. The self-dual 3-form field strength is then

$$h_3 = \frac{a}{2}\frac{R^3}{\kappa^3}\left(\sin^2\alpha \sin\beta d\alpha \wedge d\beta \wedge d\gamma + \frac{\sqrt{1+\kappa^2}}{r^3} dt \wedge dx \wedge dr\right) \tag{3.19}$$

and

$$H_3 = 2a\frac{R^3}{\kappa^3}\left(\frac{1}{1-a^2}\sin^2\alpha \sin\beta d\alpha \wedge d\beta \wedge d\gamma + \frac{\sqrt{1+\kappa^2}}{(1+a^2)r^3} dt \wedge dx \wedge dr\right) \tag{3.20}$$

For the scalar equation of motion, it is more involved. In our case, we have

$$\begin{aligned} &\mathcal{E}^{\underline{0}}_{\underline{t}} = \frac{R}{\kappa r}, \quad \mathcal{E}^{\underline{1}}_{\underline{r}} = \frac{R}{r}, \quad \mathcal{E}^{\underline{2}}_{\underline{x}} = \frac{R}{\kappa r}, \quad \mathcal{E}^{\underline{3}}_{\underline{r}} = \frac{R}{\kappa r}, \\ &\mathcal{E}^{\underline{4}}_{\underline{\alpha}} = \frac{R}{\kappa}, \quad \mathcal{E}^{\underline{5}}_{\underline{\beta}} = \frac{R\sin\alpha}{\kappa}, \qquad \mathcal{E}^{\underline{6}}_{\underline{\gamma}} = \frac{R\sin\alpha\sin\beta}{\kappa}, \end{aligned} \tag{3.21}$$

where we have set the veilbein of AdS_7 part of the target spacetime as

$$\begin{aligned} &\hat{\theta}^0 = \frac{R}{y}dt, \quad \hat{\theta}^1 = \frac{R}{y}dy, \quad \hat{\theta}^2 = \frac{R}{y}dx, \quad \hat{\theta}^3 = \frac{R}{y}dr, \\ &\hat{\theta}^4 = \frac{Rr}{y}d\alpha, \quad \hat{\theta}^5 = \frac{Rr\sin\alpha}{y}d\beta, \qquad \hat{\theta}^6 = \frac{Rr\sin\alpha\sin\beta}{y}d\gamma. \end{aligned} \tag{3.22}$$

The corresponding spin connection could be found in appendix.

The scalar equation of motion involves

$$\nabla_m \mathcal{E}^{\underline{c}}_{\underline{n}} = \partial_m \mathcal{E}^{\underline{c}}_{\underline{n}} - \Gamma^p_{mn}\mathcal{E}^{\underline{c}}_{\underline{p}} + \mathcal{E}^{\underline{a}}_{\underline{m}}\mathcal{E}^{\underline{b}}_{\underline{n}}\omega^{\underline{c}}_{\underline{ab}} \tag{3.23}$$

where Γ^p_{mn} is the Christoffel symbol and $\omega^{\underline{c}}_{\underline{ab}}$ is the spin connection of the background spacetime. The calculation shows that

$$G^{mn}\nabla_m \mathcal{E}^{\underline{c}}_n = 0, \qquad \text{except } \underline{c} = \underline{1} \text{ or } \underline{3}. \tag{3.24}$$

The nontrivial components come from $\underline{c} = \underline{1}$ or $\underline{3}$. The right hand side of the scalar equation of motion consists of the matrix $P^{\underline{c}}_{\underline{a}} = \delta^{\underline{c}}_{\underline{a}} - \mathcal{E}^m_{\underline{a}}\mathcal{E}_m{}^{\underline{c}}$, which has nonvanishing components

$$P_{\underline{a}}{}^{\underline{c}} = \begin{pmatrix} \frac{1}{1+\kappa^2} & -\frac{\kappa}{1+\kappa^2} \\ -\frac{\kappa}{1+\kappa^2} & \frac{\kappa^2}{1+\kappa^2} \end{pmatrix}. \tag{3.25}$$

where $\underline{a}, \underline{c}$ take values $\underline{1}, \underline{3}$.

For the background flux, we have a dual 7-form field strength in AdS_7 part,

$$H_{\underline{01}\cdots\underline{6}} = \frac{6}{R} \tag{3.26}$$

Note that our convention is a little different from the literature by a factor 2 since we have rescaled the radius of AdS_7.

On the right hand side of the scalar equation, only 7-form field strength contributes since the M5-brane worldvolume is embedded simply into AdS_7 and there is no induced 4-form field strength on it.

It turns out that the nontrivial components $\underline{c} = \underline{1}$ and $\underline{3}$ of the scalar equation of motion give the same constraint:

$$\frac{(1+a^2)^2}{1+\kappa^2} + (1-a^2)^2 = -2\frac{1-a^4}{\sqrt{1+\kappa^2}}. \tag{3.27}$$

This equation could be solved and gives the relation

$$a = \frac{\pm\kappa}{\sqrt{1+\kappa^2}-1} \tag{3.28}$$

In short, we have obtained a string soliton solution of M5-brane, once the relation (3.28) is satisfied. We will show in section 5 that our solution is indeed half-BPS. This solution matches with the one found in section 2.3 in [26].

Let us consider some properties of this string soliton solution. One can calculate the charges of this string soliton. Since our solution could be taken as M2-branes ending on M5-brane, with M2-brane worldvolume extending along t, x, r, the charges could be calculated by

$$Q_E = \frac{1}{\mathrm{Vol}(S^3)}\int_{S^3} \star H \tag{3.29}$$

$$Q_M = \frac{1}{\mathrm{Vol}(S^3)}\int_{S^3} H \tag{3.30}$$

where S^3 is the transverse S^3 and $\star$ here means the Hodge dual with respect to the metric of the M5-brane worldvolume without the string soliton. Here is a subtlety. If we take the

strategy suggested in [29] and think that the metric of the 5-brane worldvolume without string soliton is just

$$ds^2 = \frac{R^2}{\kappa^2 r^2}(-dt^2 + dx^2 + dr^2) + \frac{R^2}{\kappa^2}(d\alpha^2 + \sin^2\alpha d\beta^2 + \sin^2\alpha \sin^2\beta d\gamma^2) \tag{3.31}$$

which indicates that the worldvolume is a $AdS_3 \times S^3$ with the same radius, then our solution has opposite electric and magnetic charge:

$$Q_E = \pm\frac{R^3}{l_p^3\kappa^2} \tag{3.32}$$

$$Q_M = \mp\frac{R^3}{l_p^3\kappa^2}. \tag{3.33}$$

However, the above treatment could be problematic. When we turn off the charge, the S^3 part shrinks also so we have no M5-brane worldvolume anymore. This means that the solution is not the same self-dual string soliton on M5-brane as the one discussed in [29]. It is more like the case in [6], where a D3-brane is blown up by the Wilson line. Analogously, it would be better to calculate the charge from the action itself. For the magnetic charge, the above result is fine. But for the electric charge, it could be better to start from the conjugate momentum of the 2-form gauge potential, which is defined to be

$$\Pi = 2\frac{\delta\mathcal{L}}{\delta(\partial_t A_{xr})}. \tag{3.34}$$

In the Wilson loop case, the conjugate momentum to the gauge potential gives the charge of F1-strings. We expect that the conjugate momentum of the 2-form gauge potential gives the electric charge of membranes. Let us first start from the non-chiral action. Using the nonlinear self-duality relation (2.18), we have

$$\Pi = *H, \tag{3.35}$$

where $*$ is respect to the induced metric of M5-brane. So, in this sense, the electric charge of the Wilson surface is

$$Q_E = \pm\frac{R^3\sqrt{1+\kappa^2}}{l_p^3\kappa^2}, \tag{3.36}$$

which is different from the magnetic charge.

However, if we start from the covariant action (2.20), then the conjugate momentum is not so simple, it is

$$\Pi' = \left(\frac{\sqrt{-g}}{\sqrt{-\det(g_{mn} + i\tilde{H}_{mn})}} - 1\right) * H, \tag{3.37}$$

which in this case gives

$$Q'_E = -\left(1 + \frac{1}{\sqrt{1+\kappa^2}}\right)Q_E. \tag{3.38}$$

Therefore, we have shown that the conjugate momentum of the gauge potential depends on the choice of the action. We are not certain which one we should use. Fortunately, the magnetic charge is always well-defined. It characterize the winding number of membranes ending on the M5-brane. We will use it in our following discussion.

It is remarkable that the sign in (3.28) has physical implication. From the discussion on the charge, we know that Q_M is proportional to $-a$. So the minus sign in (3.28) means that we have membranes ending on the M5-brane, while the plus sign in (3.28) indicates anti-membranes on M5-brane. We will see that both cases the soliton solution is half-supersymmetric.

It is also interesting to consider the bulk action in different formalism. From the nonchiral action (2.28), the action of M5 brane in this case is

$$S = T_5 \int dtdxdrd\alpha d\beta d\gamma \left(\frac{R}{\kappa r}\right)^3 \left(\frac{R}{\kappa}\right)^3 \sin^2\alpha \sin\beta \left(\frac{\kappa^2}{2}\right).$$

The integral over r in the action shows that it is quadratically divergent and is proportional to the area of the Wilson surface:

$$S = \frac{N|Q_M|}{4\pi^2} TX \frac{1}{y_0^2}, \tag{3.39}$$

where T and X indicates the integral over t and x, and y_0 is a cut-off. Compared with the result from the membrane calculation (3.6) , we find that basically they differs by a Q_M factor. This fact indicates that for a Wilson surface operator in the symmetric representation its expectation value is Q_M times the one of the fundamental representation. Recall that Q_M is the charge of the membrane and should be identified with the rank of the representation. This is very similar to the Wilson line case. However, there is a difference besides Q_M in the prefactor. It could be absorbed in the cutoff. Or it indicates that these quadratic divergence should be cancelled by appropriate counter terms, considering the BPS nature of the configuration which will be shown in section 5.

Unlike the infinite straight Wilson line case, the action of M5-brane is not vanishing, but is proportional to the charge. This is not strange since our soliton solution is a self-dual one, with both electric and magnetic charge. Even in the BPS Wilson-t' Hooft line case, the action of D3-brane is not vanishing if not taking into account of the boundary term [13]. Our case here is very similar. The BPS nature of our solution suggests that if we take into account of the boundary term, the action could be vanishing. However the boundary terms in our case seems to be tricky. One may naively work out the conjugate momenta of y and A_{xr}. For the 2-form gauge potential, its conjugate momentum has been given as above, and actually the contribution from $\Pi^{txr} H_{txr}$ exactly cancel the bulk action. This seems indicate that one should only consider the boundary terms from conjugate momentum of gauge potential. From the following discussions on other cases, we will see that the issue is not so simple.

One could also study the action from its covariant form (2.20). Since the action involves an auxiliary field, it needs some efforts to carry it out. To simplify the calculation, one can

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choose the vector $\vec{v}$ to have nonvanishing components v_t and v_r. With this choice, one can check that the generalized self-dual condition (2.18) is satisfied and

$$\sqrt{-\det(g_{mn}+i\tilde{H}_{mn})}=\sqrt{(-g)\left(1+\frac{1}{2}\mathrm{Tr}\tilde{H}^2\right)}. \tag{3.40}$$

The straightforward calculation shows that the action of the solution is identical to the one from nonchiral action. However, one should note that since the conjugate momentum of the gauge potential from covariant action is different from the one from nonchiral action, the boundary term gives the different contribution. It is not clear which action one should use to discuss the boundary terms. It turns out for the soliton solutions studied in this paper, the two bulk actions are the same. For the boundary terms, we will just focus on the ones from the nonchiral action.

Note that for this solution, we can take M5-brane as the blow-up of M2-brane. This is reminiscent of the D3-brane description of the Wilson line studied in [6]. In the Wilson line case, if the Wilson operator belongs to the symmetric representation, its brane description is D3-brane, which has the worldvolume $AdS_2\times S^2$ embedded in AdS_5. While if the Wilson operator belongs to the antisymmetric representation, its brane description is a D5-brane, which has the worldvolume $AdS_2\times S^4$ with AdS_2 in AdS_5 and S^4 in S^5 [7]. In our case, we have a M5-brane description of the infinite straight Wilson surface. Since this M5-brane worldvolume is completely embedded in AdS_7, analogue to the Wilson line case, we may take the M5-brane solution discussed in this section correspond to the Wilson surface operator in the "symmetric representation". Intuitively, we can take the Wilson surface in the "symmetric representation" as the multi-wound Wilson surface. Later on, we will see that there is another M5-brane description of the Wilson surface in the "antisymmetric representation", where the M5-brane worldvolume is still a $AdS_3\times\tilde{S}^3$ but with $\tilde{S}^3$ in S^4.

3.2 Spherical Wilson surface

The spherical Wilson surface could be obtained from the straight Wilson surface through a conformal transformation. The spherical Wilson surface in the fundamental representation was firstly studied in [20] in the context of AdS/CFT correspondence. Unlike the straight Wilson surface, whose membrane boundary is a two-plane, the boundary of membrane for a spherical Wilson surface is a two-sphere. With the same philosophy, one can get the action of the membrane [20]

$$S=2N\left(\frac{2L^2}{\epsilon^2}-2\ln\frac{2L}{\epsilon}-1+\mathcal{O}(\epsilon)\right) \tag{3.41}$$

where L is the radius of two-sphere. It has both the quadratic and logarithmic divergences.

In order to consider the M5-brane description of the spherical Wilson surface, it is more convenient to work in the Euclidean signature as in [6] and start with the following metric of AdS_7:

$$ds^2=\frac{R^2}{y^2}(dy^2+dr_1^2+r_1^2(d\alpha^2+\sin^2\alpha d\beta^2)+dr_2^2+r_2^2(d\gamma^2+\sin^2\gamma d\delta^2). \tag{3.42}$$

The Wilson surface will be placed at $r_1 = L$ and $r_2 = 0$. Let us change the coordinates (r_1, r_2, y) to (ρ, η, θ) by the following relation:

$$r_1 = \frac{L\cos\eta}{\cosh\rho - \sinh\rho\cos\theta}, \quad r_2 = \frac{L\sinh\rho\sin\theta}{\cosh\rho - \sinh\rho\cos\theta} \ , y = \frac{L\sin\eta}{\cosh\rho - \sinh\rho\cos\theta}, \quad (3.43)$$

then we have the AdS_7 metric as

$$ds^2 = \frac{R^2}{\sin^2\eta}\Big(d\eta^2 + \cos^2\eta(d\alpha^2 + \sin^2\alpha d\beta^2) + d\rho^2 + \sinh^2\rho(d\theta^2 + \sin^2\theta d\gamma^2 + \sin^2\theta\sin^2\gamma d\delta^2)\Big) \quad (3.44)$$

Here, the coordinates take the range $\rho \in [0, \infty), \theta, \alpha, \gamma \in [0, \pi), \beta, \delta \in [0, 2\pi), \eta \in [0, \pi/2)$.

To find the appropriate M5-brane that describes the blow-up of the Wilson surface, we may take $(\rho, \alpha, \beta, \theta, \gamma, \delta)$ as the worldvolume coordinates of M5-brane and assume that η be only the function of ρ. Equivalently, we can think η instead of ρ as the worldvolume coordinate. Inspired by the solution in D3-brane of the Wilson line, we make the following ansatz between η and ρ:

$$\sin\eta = \kappa^{-1}\sinh\rho, \quad (3.45)$$

then the induced metric is

$$\begin{aligned} ds^2 = \frac{R^2}{\sin^2\eta}\Big(\frac{1+\kappa^2}{1+\kappa^2\sin^2\eta}d\eta^2 + \cos^2\eta(d\alpha^2 + \sin^2\alpha d\beta^2)\Big) \\ + R^2\kappa^2(d\theta^2 + \sin^2\theta d\gamma^2 + \sin^2\theta\sin^2\gamma d\delta^2). \end{aligned} \quad (3.46)$$

We turn on the self-dual field strength on the M5-brane:

$$h_3 = 2a\Big(i\Big(\frac{R}{\sin\eta}\Big)^3\sqrt{\frac{1+\kappa^2}{1+\kappa^2\sin^2\eta}}\cos^2\eta\sin\alpha d\eta\wedge d\alpha\wedge d\beta + R^3\kappa^3\sin^2\theta\sin\gamma d\theta\wedge d\gamma\wedge d\delta\Big). \quad (3.47)$$

Notice that due to the Euclidean signature, there is a factor i in $h_{\eta\alpha\beta}$. Similarly we can work out k^{mn}, k^2, Q and open membrane metric G^{mn}. The field strength H_3 is just

$$\begin{aligned} H_3 = 2a\Big(i\frac{1}{1+a^2}\Big(\frac{R}{\sin\eta}\Big)^3\sqrt{\frac{1+\kappa^2}{1+\kappa^2\sin^2\eta}}\cos^2\eta\sin\alpha d\eta\wedge d\alpha\wedge d\beta \\ + \frac{1}{1-a^2}R^3\kappa^3\sin^2\theta\sin\gamma d\theta\wedge d\gamma\wedge d\delta\Big). \end{aligned} \quad (3.48)$$

With these setups, let us check if they satisfy the equation of motion. The components of Levi-Civita connection of the metric (3.46) are listed in appendix. It is straightforward but tedious to check that the tensor equation is satisfied. For the scalar equation, we have

$$\begin{aligned} \mathcal{E}^{\underline{1}}_{\eta} = \frac{R}{\sin\eta}, \quad \mathcal{E}^{\underline{2}}_{\alpha} = \frac{R\cos\eta}{\sin\eta}, \quad \mathcal{E}^{\underline{3}}_{\beta} = \frac{R\cos\eta\sin\alpha}{\sin\eta}, \\ \mathcal{E}^{\underline{4}}_{\eta} = \frac{\kappa R\cos\eta}{\sin\eta\sqrt{1+\kappa^2\sin^2\eta}}, \quad \mathcal{E}^{\underline{5}}_{\theta} = \kappa R, \quad \mathcal{E}^{u6}_{\gamma} = \kappa R\sin\theta, \quad \mathcal{E}^{\underline{7}}_{\delta} = \kappa R\sin\theta\sin\gamma. \end{aligned} \quad (3.49)$$

Here the vielbein of the metric (3.44) are

$$\hat{\theta}^{\underline{1}} = \frac{R}{\sin\eta}d\eta, \qquad \hat{\theta}^{\underline{2}} = \frac{R\cos\eta}{\sin\eta}d\alpha, \qquad \hat{\theta}^{\underline{3}} = \frac{R\cos\eta\sin\alpha}{\sin\eta}d\beta, \qquad \hat{\theta}^{\underline{4}} = \frac{R}{\sin\eta}d\rho,$$
$$\hat{\theta}^{\underline{5}} = \frac{R\sinh\rho}{\sin\eta}d\theta, \quad \hat{\theta}^{\underline{6}} = \frac{R\sinh\rho\sin\theta}{\sin\eta}d\gamma, \quad \hat{\theta}^{\underline{7}} = \frac{R\sinh\rho\sin\theta\sin\gamma}{\sin\eta}d\delta. \tag{3.50}$$

From the scalar equation, we obtain one nontrivial relation coming from the cases when $\underline{c} = \underline{1}$ or $\underline{4}$:

$$\frac{\kappa}{\sqrt{1+\kappa^2}} = -\frac{1-a^2}{1+a^2}. \tag{3.51}$$

This is actually the same relation (3.28) if we change $\kappa \to \frac{1}{\kappa}$.

The charges of the string is the same as the ones in the straight Wilson surface case, once we take into account the difference of the parameter κ in two cases.

The non-chiral action gives us

$$S_{M5} = T_5 \int *K, \tag{3.52}$$

where $K = -\frac{1+a^4}{1-a^4}$.

The Wess-Zumino part of the action is more involved. The bulk 6-form gauge potential is

$$\begin{aligned} C_6 &= \left(\frac{R}{y}\right)^6 r_1^2 r_2^2 \sin\alpha\sin\gamma dr_1\wedge d\alpha\wedge d\beta\wedge dr_2\wedge d\gamma\wedge d\delta \\ &= R^6\frac{\cos^3\eta\sinh^3\rho\sin^2\theta\sin\alpha\sin\gamma}{\sin^6\eta}d\rho\wedge d\alpha\wedge d\beta\wedge d\theta\wedge d\gamma\wedge d\delta \\ &\quad -R^6\frac{\cos^2\eta\sinh^2\rho\sin^3\theta\sin\alpha\sin\gamma}{\sin^5\eta(\cosh\rho-\sinh\rho\cos\theta)}d\eta\wedge d\alpha\wedge d\beta\wedge d\rho\wedge d\gamma\wedge d\delta \\ &\quad +R^6\frac{\cos^2\eta\sinh^3\rho\sin^2\theta\sin\alpha\sin\gamma(\sinh\rho-\cos\theta\cosh\rho)}{\sin^5\eta(\cosh\rho-\sinh\rho\cos\theta)}d\eta\wedge d\alpha\wedge d\beta\wedge d\theta\wedge d\gamma\wedge d\delta \end{aligned} \tag{3.53}$$

The total bulk action of M5-brane turns out to be

$$\begin{aligned} S &= T_5(8\pi^3R^6)\left(\frac{\kappa^2}{4}\right)\left(\frac{1}{\eta_0^2}+\ln\eta_0\right) \\ &= \frac{N|Q_M|}{2\pi}\left(\frac{1}{\eta_0^2}+\ln\eta_0\right) \end{aligned} \tag{3.54}$$

where η_0 is a cutoff near 0. It is remarkable that from the form of the Wess-Zumino action there could be quartic divergence. However it turns out to be vanishing in the end. So the bulk action is actually of both quadratic and logarithmic divergences, with the similar structure as (3.41). Now $\kappa^2 = (8\pi N)^{-1}Q_M$, which is very small in the large N limit. Here we have made the conformal transformation so the radius of the sphere does not appear in the above expression. To compare with the existing result in the literature, we can replace

$1/\eta_0$ with L/η_0 in the above relation, where L is the radius of sphere. It is remarkable that the bulk action is linear in the charge Q_M of Wilson surface. This is consistent with the result from the field theory calculation [20]. The divergence above has two origins, one is from the conformal anomaly which relates the straight Wilson surface to the spherical one, the other is from the divergence in the original straight Wilson surface. Since the boundary of Wilson surface is 1-dimensional, it will not induce any conformal anomaly.

One may wonder the boundary terms also contribute to the action. Especially one may wonder if the contribution from gauge potential part could cancel the above divergence exactly. Unfortunately it is not the case anymore. After taking into account of it, the nonchiral action is still logarithmically divergent:

$$S_{M5} + \Pi^{txr} H_{txr} \sim \frac{\kappa^4}{4} \ln \eta_0. \tag{3.55}$$

Since κ^2 is very small, the above contribution is next leading order result. In other words, the leading order divergent term is actually cancelled by the boundary term.

It is remarkable that unlike the Wilson loop case, there is no finite contribution from the integral directly, no matter if or not we take into account of the boundary terms.

Therefore we have a different story on Wilson surface from Wilson line. In the Wilson line case, the expectation of the straight Wilson line is vanishing and the one of the circular Wilson line get the contribution from the conformal anomaly of the boundary. In the Wilson surface case, the expectation values of both kinds of Wilson surfaces are not vanishing.

4. M5-brane description of the Wilson surface in the antisymmetric representation

In the study of the brane picture of the Wilson-loop, one knows that for the Wilson loop in the anti-symmetric representation it should be described by D5-brane whose worldvolume is of topology $AdS_2 \times S^4$ with AdS_2 being in AdS_5 and S^4 in S^5 [7]. In the case of the Wilson surface, one may expect that there exit another M5-brane description. We will show in this section this is true. We find that though this M5-brane is of the same topology $AdS_3 \times \tilde{S}^3$, unlike the case we studied in the above sections, the $\tilde{S}^3$ part is embedded in S^4.

4.1 Straight Wilson surface

Let the worldvolume coordinates of M5-branes be ξ_i, $i = 0, \ldots 5$ and the embedding be

$$\begin{aligned} \xi_0 = t, \quad \xi_1 = x, \quad \xi_2 = y, \\ \xi_3 = \zeta_2, \quad \xi_4 = \zeta_3, \quad \xi_5 = \zeta_4, \quad \zeta_1 = \zeta^0 \end{aligned} \tag{4.1}$$

where ζ_i are the angular coordinates of S^4. Here we let ζ_1 be fixed at a constant ζ^0. The induced metric is

$$ds^2_{\rm ind} = \frac{R^2}{y^2}(-dt^2 + dx^2 + dy^2) + \frac{R^2 \sin^2 \zeta^0}{4}(d\zeta_2^2 + \sin^2 \zeta_2 d\zeta_3^2 + \sin^2 \zeta_2 \sin^2 \zeta_3 d\zeta_4^2). \tag{4.2}$$

In this case, we take the self-dual 3-form field strength on the M5-brane worldvolume to be

$$h_3 = 2aR^3\left(\frac{1}{y^3}dt\wedge dx\wedge dy + \frac{\sin^3\zeta^0}{8}\sin^2\zeta_2\sin\zeta_3 d\zeta_2\wedge d\zeta_3\wedge d\zeta_4\right) \tag{4.3}$$

Similar to the above cases, we can get k^{mn}, $k^2 = \frac{3}{2}a^4$ and $Q = 1-a^4$. The open membrane metric G^{mn} take the diagonal form:

$$\begin{aligned} G^{tt} &= -G^{xx} = -G^{yy} = (1+a^2)^2\left(\frac{y}{R}\right)^2, \\ G^{22} &= (1-a^2)^2\frac{4}{R^2\sin^2\zeta^0},\quad G^{33} = \frac{G^{22}}{\sin^2\zeta_2},\quad G^{44} = \frac{G^{22}}{\sin^2\zeta_2\sin^2\zeta_3}, \end{aligned} \tag{4.4}$$

where G^{ii} denotes $G^{\zeta_i\zeta_i}$. And the physical 3-form is

$$H_3 = 2aR^3\left(\frac{1}{(1+a^2)y^3}dt\wedge dx\wedge dy + \frac{\sin^3\zeta^0}{8(1-a^2)}\sin^2\zeta_2\sin\zeta_3 d\zeta_2\wedge d\zeta_3\wedge d\zeta_4\right), \tag{4.5}$$

satisfying $dH_3 = 0$.

It is straightforward to check if it is possible and under what condition if possible that the above ansatz satisfy the equations of motion. The tensor equation holds under the above setup. For the scalar equation, the AdS_3 part is trivially satisfied. For the $\tilde{S}^3$ part, we have

$$\mathcal{E}^{\underline{2}}_{\zeta_2} = \frac{R\sin\zeta^0}{2},\quad \mathcal{E}^{\underline{3}}_{\zeta_3} = \frac{R\sin\zeta^0\sin\zeta_2}{2},\quad \mathcal{E}^{\underline{4}}_{\zeta_4} = \frac{R\sin\zeta^0\sin\zeta_2\sin\zeta_3}{2}, \tag{4.6}$$

where we set the vielbein of S^4 part to be

$$\hat{\theta}^{\underline{1}} = \frac{R}{2}d\zeta_1,\quad \hat{\theta}^{\underline{2}} = \frac{R}{2}\sin\zeta_1 d\zeta_2,\quad \hat{\theta}^{\underline{3}} = \frac{R}{2}\sin\zeta_1\sin\zeta_2 d\zeta_3,\quad \hat{\theta}^{\underline{4}} = \frac{R}{2}\sin\zeta_1\sin\zeta_2\sin\zeta_3 d\zeta_4 \tag{4.7}$$

We list the relevant Christoffel symbol with respect to (4.2) and the spin connection with respect to (4.7) in appendix.

The nontrivial relation for the scalar equation comes from $\underline{c} = \underline{1}$. Here the left hand side of the equation is not vanishing due to the nonvanishing contribution from the spin connection. And on the right hand side, since $H_{\underline{1234}} = \frac{6}{R}$, it gives nonvanishing contribution. This leads to a relation

$$1-a^2 = -2\frac{a\sin\zeta^0}{\cos\zeta^0} \tag{4.8}$$

or

$$a = \frac{\pm 1+\sin\zeta^0}{\cos\zeta^0}. \tag{4.9}$$

Therefore, we have obtained another M5-brane soliton solution once (4.9) is satisfied. This solution is the same one in section 2.2 in [26], discussed in PST formalism.

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Let us calculate the charges of the Wilson surface. The magnetic charge is easy to obtain:

$$Q_M = -\frac{R^3 \sin^2\zeta^0 \cos\zeta^0}{8l_p^3}. \tag{4.10}$$

For the electric charge, it is much subtler. One may define it from (3.29), which gives you

$$Q_E = \frac{R^3 \sin^3\zeta^0 \cos\zeta^0}{8l_p^3}. \tag{4.11}$$

On the other hand, one can define the electric charge from the conjugate momentum. In this case, the conjugate momentum not only get contribution from the non-chiral action, but also from the Wess-Zumino part. And in the Wess-Zumino part of the action, there exist an ambiguity in defining the 3-form gauge potential. We make the following choice

$$C_3 = -\frac{3}{8}R^3\Big(-\cos\zeta^0 + \frac{1}{3}\cos^3\zeta^0\Big)\sin^2\zeta_2 \sin\zeta_3 . d\zeta_2 \wedge d\zeta_3 \wedge d\zeta_4 \tag{4.12}$$

Then the electric charge is

$$Q_E = \frac{R^3}{16l_p^3}\cos\zeta^0\big(\sin^3\zeta^0 + 3 - \cos^2\zeta^0\big). \tag{4.13}$$

Similar to the case in section 3, the sign in (4.9) is physical. The situation here is a little subtler. No matter which sign we take, we always get the same formulae on magnetic and electric charges. In other words, what kind of membrane the M5-brane feels depends on $\cos\zeta^0$ rather than the sign in (4.9). Nevertheless, we will show that the different choice of sign indicates the different supersymmetries the soliton solution keeps.

Note that once we turn off the 3-form field on M5-brane, there still exists an M5-brane solution, which reside at $\zeta^0 = \pi/2$. This means that the M5-brane without flux could be embedded in the background without instability.

The bulk action is

$$\begin{aligned} S &= T_5 \int (*K - 2\underline{C}_3 \wedge H_3) \\ &= \frac{T_5}{8} 4\pi^2 R^6 T X \frac{1}{2y_0^2} \\ &= \frac{N^2}{2\pi} T X \frac{1}{y_0^2}, \end{aligned} \tag{4.14}$$

which is quadratically divergent, and similar to (3.6). Now since $Q_M \sim N$, the action is still proportional to NQ_M. Similarly, if we try to take the contribution from boundary terms coming from the conjugate momentum of gauge potential into account, we have

$$S \sim \sin^2\zeta^0 \frac{1}{y_0^2}. \tag{4.15}$$

4.2 Spherical Wilson surface

For the spherical Wilson surface, we have to do a conformal transformation of the above one. The embedding of $\tilde{S}^3$ in S^4 is the same as before. For the AdS_3 part, it is somehow different. The metric of Euclideanized AdS_7 take the form (3.42). Let us assume that the spherical Wilson surface satisfy $r_1^2 + y^2 = L^2$, namely a sphere S^2 with radius L, and later on we will check such kind of embedding satisfies the equations of motion. Let

$$y = L\cos\delta, \quad r_1 = L\sin\delta, \tag{4.16}$$

then the AdS_3 part of the induced metric of M5-brane is

$$ds^2_{\text{ind}} = \frac{R^2}{\cos^2\delta}(d\delta^2 + \sin^2\delta(d\alpha^2 + \sin^2\alpha d\beta^2)) \tag{4.17}$$

The self-dual 3-form field strength on the M5-brane worldvolume could be set to

$$h_3 = 2aR^3\left(i\frac{\sin^2\delta\sin\alpha}{\cos^3\delta}d\delta\wedge d\alpha\wedge d\beta + \frac{\sin^3\zeta^0}{8}\sin^2\zeta_2\sin\zeta_3 d\zeta_2\wedge d\zeta_3\wedge d\zeta_4\right). \tag{4.18}$$

From it, we can calculate the other quantities as before. The only differences from the straight case are

$$G^{\delta\delta} = (1+a^2)^2\left(\frac{\cos\delta}{R}\right)^2, \quad G^{\alpha\alpha} = \frac{G^{\delta\delta}}{\sin^2\delta}, \quad G^{\beta\beta} = \frac{G^{\delta\delta}}{\sin^2\delta\sin^2\alpha}. \tag{4.19}$$

The physical 3-form field strength is now

$$H_3 = 2aR^3\left(i\frac{1}{1+a^2}\frac{\sin^2\delta\sin\alpha}{\cos^3\delta}d\delta\wedge d\alpha\wedge d\beta + \frac{1}{1-a^2}\frac{\sin^3\zeta^0}{8}\sin^2\zeta_2\sin\zeta_3 d\zeta_2\wedge d\zeta_3\wedge d\zeta_4\right). \tag{4.20}$$

For the scalar equation, the discussion on $\tilde{S}^3$ part does not change and we find the same relation as (4.8). We needs to check if the Euclideanized AdS_3 part does not give anything nontrivial. This could be checked explicitly. Now we have

$$\mathcal{E}^{\underline{1}}_{\delta} = -\frac{R\sin\delta}{\cos\delta}, \quad \mathcal{E}^{\underline{2}}_{\delta} = R, \quad \mathcal{E}^{\underline{3}}_{\alpha} = \frac{R\sin\delta}{\cos\delta}, \quad \mathcal{E}^{\underline{4}}_{\beta} = \frac{R\sin\delta}{\cos\delta}\sin\alpha, \tag{4.21}$$

where we have set the relevant vierbeins to be

$$\hat{\theta}^{\underline{1}} = \frac{R}{y}dy, \quad \hat{\theta}^{\underline{2}} = \frac{R}{y}dr_1, \quad \hat{\theta}^{\underline{3}} = \frac{R}{y}r_1 d\alpha, \quad \hat{\theta}^{\underline{4}} = \frac{R}{y}r_1\sin\alpha d\beta. \tag{4.22}$$

Here we abuse the indices which we wish would not bring any confusion to the reader. From the embedding, it is not obvious that $\nabla_m \mathcal{E}^{\underline{c}}_m = 0$. However the explicit calculation shows that this is indeed true. The relevant Levi-Civita connection and the spin connection are put into appendix.

Unlike the case discussed in section 3, the conformal transformation from straight surface to sphere is somehow trivial. Therefore the charges of the membrane on M5-brane is the same as (4.10), (4.13). The bulk action reads

$$\begin{aligned} S &= 2\pi^3 R^6 T_5 \int_0^{\frac{\pi}{2}} \frac{\sin^2\delta}{\cos^3\delta} \\ &= \frac{\pi^3}{2} R^6 T_5 \left(\frac{2}{\epsilon^2} - \ln\frac{2}{\epsilon}\right) \\ &= N^2 \left(\frac{2}{\epsilon^2} - \ln\frac{2}{\epsilon}\right), \end{aligned} \tag{4.23}$$

where ϵ is the cutoff near $\delta = \frac{\pi}{2}$. The terms in the bracket of (4.23) looks familiar. Actually, in the above discussion we have chose the angular coordinates so that the radius of the sphere does not show up. It is easily to recover it by replace $1/\epsilon$ with L/ϵ. Then we have the similar divergent terms as (3.41). The significant difference is that the M5-brane action is proportional to the membrane charge Q_M.

5. Supersymmetry analysis

Let us check if the above solutions are supersymmetric. Firstly we need to work out the Killing spinor of the bulk background. From the discussions above, we notice that all the solution has a global symmetry $\mathrm{SO}(2,2) \times \mathrm{SO}(4) \times \mathrm{SO}(4)$. In order to make the analysis simpler, we first rewrite $AdS_7 \times S^4$ metric in form of $AdS_3 \times S^3 \times \tilde{S}^3$ fibred over two-dimensional base space:

$$ds^2 = R^2 \left(\cosh^2\rho\, ds^2_{AdS_3} + \sinh^2\rho\, d\Omega_3^2 + d\rho^2\right) + \frac{R^2}{4}\left(d\zeta_1^2 + \sin^2\zeta_1\, d\tilde{\Omega}_3^2\right), \tag{5.1}$$

In the new fibred coordinates, the 4-form flux can be written as:

$$H_4 = \frac{6}{R} e^7 \wedge e^8 \wedge e^9 \wedge e^{10}. \tag{5.2}$$

Here we use $ds^2_{AdS_3}, d\Omega_3^2, d\tilde{\Omega}_3^2$ to denote the metric of unit AdS_3 and two unit S^3's, respectively. The e^M's, $M = 0, \ldots, 10$ are the vielbein of this metric.

We use Γ^M to denote the 11-dimensional Gamma matrices. They can be written as the following form [39]:

$$\begin{aligned} \Gamma^0 &= \gamma_8 \otimes \check{\sigma}^0 \otimes 1 \otimes 1 \otimes \sigma_1, & \Gamma^1 &= \gamma_8 \otimes \check{\sigma}^1 \otimes 1 \otimes 1 \otimes \sigma_1, \\ \Gamma^2 &= \gamma_8 \otimes \check{\sigma}^2 \otimes 1 \otimes 1 \otimes \sigma_1, & \Gamma^3 &= \gamma_8 \otimes 1 \otimes \hat{\sigma}^3 \otimes 1 \otimes \sigma_2, \\ \Gamma^4 &= \gamma_8 \otimes 1 \otimes \hat{\sigma}^4 \otimes 1 \otimes \sigma_2, & \Gamma^5 &= \gamma_8 \otimes 1 \otimes \hat{\sigma}^5 \otimes 1 \otimes \sigma_2, \\ \Gamma^6 &= \gamma^6 \otimes 1 \otimes 1 \otimes 1 \otimes 1, & \Gamma^7 &= \gamma^7 \otimes 1 \otimes 1 \otimes 1 \otimes 1, \\ \Gamma^8 &= \gamma_8 \otimes 1 \otimes 1 \otimes \tilde{\sigma}^8 \otimes \sigma_3, & \Gamma^9 &= \gamma_8 \otimes 1 \otimes 1 \otimes \tilde{\sigma}^9 \otimes \sigma_3, \\ \Gamma^{10} &= \gamma_8 \otimes 1 \otimes 1 \otimes \tilde{\sigma}^{10} \otimes \sigma_3. \end{aligned} \tag{5.3}$$

Here $(\gamma^6, \gamma^7, \gamma_8), (\check{\sigma}^1, \check{\sigma}^2, \check{\sigma}^3), (\hat{\sigma}^3, \hat{\sigma}^4, \hat{\sigma}^5), (\tilde{\sigma}^8, \tilde{\sigma}^9, \tilde{\sigma}^{10}), (\sigma_1, \sigma_2, \sigma_3)$ are five sets of Pauli matrices and $\check{\sigma}^0 = i\check{\sigma}^3$.

The first step is to find the Killing spinor in $AdS_7 \times S^4$ with the above fibred coordinates. In order to do so, we need to use the following Killing spinors of the unit AdS_3 and the two unit S^3's:

$$\mathring{\nabla}_p \check{\chi}^I_{a'} = \frac{i}{2} a' \check{\sigma}_p \check{\chi}^I_{a'}, \ (p = 0, 1, 2, a' = \pm 1, I = 1, 2), \tag{5.4}$$

$$\mathring{\nabla}_p \hat{\chi}^J_{b'} = \frac{1}{2} b' \hat{\sigma}_p \hat{\chi}^J_{b'}, \ (p = 3, 4, 5, b' = \pm 1, J = 1, 2), \tag{5.5}$$

$$\mathring{\nabla}_p \chi^K_{c'} = \frac{i}{2} c' \sigma_p \chi^K_{c'}, \ (p = 8, 9, 10, c' = \pm 1, K = 1, 2), \tag{5.6}$$

and decompose the 11-dimensional spinor ξ as

$$\xi = \sum_{a'b'c'IJK} \epsilon_{a'b'c'IJK} \check{\chi}^I_{a'} \otimes \hat{\chi}^J_{b'} \otimes \chi^K_{c'}, \tag{5.7}$$

where each $\epsilon_{a'b'c'IJK}$ is a pair of 2-dimensional spinors with $\gamma^6, \gamma^7, \gamma_8, \sigma_1, \sigma_2, \sigma_3$ acting on it. In another word, each $\epsilon_{a'b'c'IJK}$ belongs to the tensor product of the space of the 2-dimensional spinor and $\mathbf{C}^2$, and $\gamma^6, \gamma^7, \gamma_8$ act on the space of the 2-dimensional spinor, while $\sigma_1, \sigma_2, \sigma_3$ act on $\mathbf{C}^2$.

From the Killing spinor equation and the above decomposition, we obtain the following supersymmetric conditions:

$$(a'\sigma_1 + i \sinh\rho\gamma^7 - i\cosh\rho\gamma^7\sigma_3)\epsilon = 0, \tag{5.8}$$

$$(b'\sigma_2 + \cosh\rho\gamma^7 - \sinh\rho\gamma^7\sigma_3)\epsilon = 0, \tag{5.9}$$

$$(c'\sigma_3 - \cos\zeta_1\gamma^6 + \sin\zeta_1\gamma^7\sigma_3)\epsilon = 0, \tag{5.10}$$

$$\frac{\partial\epsilon}{\partial\rho} = \frac{1}{2}\sigma_3\,\epsilon, \quad \frac{\partial\epsilon}{\partial\zeta_1} = \frac{i}{2}\gamma_8\sigma_3\,\epsilon. \tag{5.11}$$

Now we begin to solve these equations. Using eq. (5.11), we find that ϵ can be written as

$$\epsilon = \exp\left(\frac{1}{2}\sigma_3\rho + \frac{i}{2}\gamma_8\sigma_3\zeta_1\right)\zeta. \tag{5.12}$$

Here ζ is a pair of constant spinors. Then eqs. (5.8)–(5.10) lead to the following projective conditions:

$$a'\gamma^7\sigma_2\zeta = \zeta, \quad b'\gamma^7\sigma_2\zeta = -\zeta, \quad c'\gamma^6\sigma_3\zeta = \zeta, \tag{5.13}$$

Since for each a', b', c', I, J, K, $\epsilon_{a'b'c'IJK}$ is a pair of two dimensional spinors and $a', b', c' = \pm 1, I, J, K = 1, 2$, totally there are 2^8 components. After imposing 3 projection conditions each of which project out half of the components, we have 2^5 complex components. Finally imposing Majorana condition leaves 2^5 real components.

Now let us introduce the Gamma matrix Γ_{M5}, which is determined by the M5-brane worldvolume and the flux on it [28]:

$$\Gamma_{M5} = \frac{1}{6!\sqrt{-g}}\epsilon^{j_1\cdots j_6}[\Gamma_{<j_1\cdots j_6>} + 40\Gamma_{<j_1j_2j_3>}h_{j_4j_5j_6}]. \tag{5.14}$$

Here g is the determinant of the induced worldvolume metric component, $h_{j_4j_5j_6}$ is the self-dual 3-form on the M5-brane. And $\Gamma_{<j_1\cdots j_n>}$ is defined as

$$\Gamma_{<j_1,\ldots j_n>} = \mathcal{E}^{\underline{a}_1}_{j_1}\cdots\mathcal{E}^{\underline{a}_n}_{j_n}\Gamma_{\underline{a}_1\cdots\underline{a}_n}, \tag{5.15}$$

where $\Gamma_{\underline{a}_1\cdots\underline{a}_n}$ is the product of the Gamma matrices in the frame.

The kappa symmetry projection condition is

$$\Gamma_{M5}\xi = \xi. \tag{5.16}$$

The amount of unbroken supersymmetry is determined by the solution of above equation.

For the straight Wilson surface case, the metric of the $M5$-brane and the self-dual 3-form flux on it can be written in the fibred coordinates as

$$ds^2 = R^2(\cosh^2\rho_k ds^2_{AdS_3} + \sinh^2\rho_k d\Omega_3^2), \tag{5.17}$$

$$\begin{aligned} h_3 &= \frac{a}{2}(e^0\wedge e^1\wedge e^2 + e^3\wedge e^4\wedge e^5) \\ &= \frac{a}{2}R^3\left(\cosh^3\rho_k\cosh\tilde{\rho}\sinh\tilde{\rho}d\tau\wedge d\tilde{\rho}\wedge d\tilde{\theta} + \sinh^3\rho_k\sin^2\alpha\sin\beta d\alpha\wedge d\beta\wedge d\gamma\right) \end{aligned} \tag{5.18}$$

From this, after some short calculations, one get

$$\Gamma_{M5} = -(\Gamma_{01\cdots5} + a(\Gamma_{012} - \Gamma_{345})) \tag{5.19}$$

Using the above representation of Γ^μ, one can find that the condition (5.16) is equivalent to

$$-\sigma_3\,\epsilon + a\gamma_8\sigma_1\,\epsilon + ia\gamma_8\sigma_2\,\epsilon = \epsilon. \tag{5.20}$$

When $\zeta_1 = 0$, we have $\epsilon = \exp(\frac{1}{2}\sigma_3\rho_k)\zeta$, then the above equation is equivalent to

$$-\sigma_3\zeta - \zeta + ae^{-\rho_k}\gamma_8\sigma_1\zeta + iae^{-\rho_k}\gamma_8\sigma_2\zeta = 0. \tag{5.21}$$

From this, we can obtain the following supersymmetry condition for a, ρ_k and ζ:

$$a = \pm e^{\rho_k}, \quad \pm\,\gamma_8\sigma_1\zeta = \zeta. \tag{5.22}$$

The projection conditions on ζ here are compatible with the projection conditions in eqs. (5.13) for the Killing spinors. So the supersymmetry conditions are satisfied by half of the components of the Killing spinors. In another word, our solution is half-BPS. In the case of $\zeta_1 = \pi$, we can similarly obtain the following supersymmetry conditions:

$$a = \pm e^{\rho_k}, \quad \mp\,\gamma_8\sigma_1\zeta = \zeta. \tag{5.23}$$

Let us set $\sinh\rho_k = \frac{1}{\kappa}$ to recover the induced metric (3.18) from fibred metric (5.1). Then the relation (3.28) is exactly the relation $a = \pm e^{\rho_k}$. This shows that our M5-brane soliton solution corresponding to the straight Wilson surface is half-BPS. For the spherical solution, we get the same conclusion.

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For the case of $\tilde{S}^3$, we can similarly obtain,

$$\Gamma_{M5} = -(\Gamma_{01289(10)} + a(\Gamma_{012} - \Gamma_{89(10)})). \tag{5.24}$$

Using eq. (5.3), we find that the relation (5.16) is equivalent to

$$\sigma_2\,\epsilon + a\gamma_8\sigma_1\,\epsilon + ia\gamma_8\sigma_3\,\epsilon = \epsilon \tag{5.25}$$

for the straight Wilson surface. Now, we have $\rho = 0$, then $\epsilon = \exp(\frac{i}{2}\gamma_8\sigma_3\zeta^0)\zeta$, we can find that the above equation is equivalent to

$$\begin{aligned}&\left(\cos\frac{\zeta^0}{2} + a\sin\frac{\zeta^0}{2}\right)\sigma_2\zeta + \left(a\cos\frac{\zeta^0}{2} - \sin\frac{\zeta^0}{2}\right)\gamma_8\sigma_1\zeta\\ &-\left(\cos\frac{\zeta^0}{2} + a\sin\frac{\zeta^0}{2}\right)\zeta - i\left(\sin\frac{\zeta^0}{2} - a\cos\frac{\zeta^0}{2}\right)\gamma_8\sigma_3\zeta\ = 0\end{aligned} \tag{5.26}$$

This gives us the following supersymmetry conditions on a, ζ^0 and ζ,

$$a = \frac{\pm 1 + \sin\zeta^0}{\cos\zeta^0}, \quad \pm\,\gamma_8\sigma_1\zeta = \zeta. \tag{5.27}$$

The first relation is exactly (4.9). As in the previous case, the projection condition here are also compatible with the projection conditions eqs. (5.13). Then we have shown that our solution in this case is half-BPS as well. The discussion on the spherical Wilson surface is similar.

From the above discussion, we come to the conclusion that all our solutions are half-BPS.

6. Conclusion and discussion

In this paper, we investigated the M5-brane soliton solutions in $AdS_7 \times S^4$ background. Starting from the covariant equations of motion of M5-brane, we found two classes of solutions, both having $AdS_3 \times S^3$ topology. The AdS_3 part is always in AdS_7 but S^3 could be in AdS_7 or S^4. The two different configurations give the description of the Wilson surface operators in the symmetric and the anti-symmetric representation respectively. We discussed the properties of these solutions and their implications to the Wilson surface operators from AdS/CFT correspondence. Unfortunately due to the shortage of the discussion on the Wilson surface operators in six-dimensional (2,0)-theory side, we were not able to make comparison more precisely.

From the dictionary of AdS/CFT correspondence, the exponential of bulk M5-brane action with boundary terms could give the expectation values of the surface operators. We are not certain of the boundary terms, which involves the conjugate momenta of the gauge potential and coordinate. Nevertheless, there are a few remarkable points on the bulk actions. Firstly for the straight Wilson surface operators, the bulk actions are quadratically divergent, and for the spherical ones, the bulk actions are both quadratically and logarithmically divergent. These two cases are related to each other by conformal transformation, being in consistence with argument from conformal anomaly [22]. Secondly,

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compared to the result on the Wilson surface operators in the fundamental representation from the membrane approach, the bulk M5-brane action is Q_M times the membrane action up to a numerical factor. The Q_M characterizes the charges carried by the membrane and also the rank of the representation. This fact implies that whatever the representation the Wilson surface operators are in, the possible M5-brane action should have the same structure. Namely the actions take the similar form as (3.39), (4.14) for the straight surfaces and (3.54), (4.23) for the spherical surfaces, being of the divergent terms times the rank of the representation. Thirdly, the fact that the solutions we found are all supersymmetric indicates that the Wilson surface operators are supersymmetric too. This implies that their expectation values should be exactly one since the bulk action should be vanishing after taking into account the appropriate boundary terms. Furthermore, since there is no conformal anomaly from boundary terms in our cases, the implication should make sense both in the straight and the spherical cases.

Our solutions are the examples of M5-brane self-dual string soliton solutions in curved spacetime. These solutions have been discussed in [26] from another approach. To our knowledge, the string soliton solutions in curved spacetime have not been studied carefully in the literature. The study of these soliton solution would be quite valuable and open a new window to the study of M5-brane physics and M-theory. To find more string soliton solution in curved spacetime and study their properties is an interesting question.

On the other hand, it would be very nice to understand the string soliton configurations from the dynamics of nonabelian membranes. In the Wilson loop case, there exist a dielectric description of F1(D1) blowing up to higher dimensional D-brane [12]. One may wonder if the same story is true here. However, we have no good understanding of nonabelian membrane action. In [40], a generalized Nahm equation has been proposed and the funnel solution from membrane has been constructed. But it is still an open issue how to construct the nonabelian membrane action, even in the flat spacetime [41]. In the case at hand, we need to know the nonabelian action in curved spacetime with background flux. We expect that some kind of Myers effect [11] exists in M-theory. This is a very important question.

The six-dimensional (2,0) superconformal field theory is very nontrivial. Some people have proposed the DLCQ matrix description of the theory [19]. It would be nice to see if this description could address the Wilson surface operators issue. This may help us to make the AdS/CFT dictionary in this case more precise.

The string soliton solutions constructed in this paper are half-BPS. It would be interesting to find other string soliton solutions with less supersymmetry. These string soliton solutions will correspond to the membranes ending on M5-branes in $AdS_7 \times S^4$ background. For the discussion on such soliton solutions in flat spacetime, see [42].

There are several subtleties in our discussion. We are not satisfied with the boundary terms we discussed. The main trouble comes from the ambiguity in choosing the action. Unlike the DBI action for D-brane in string theory, there is no well-accepted action for M5-brane. The different action may lead to different conjugate momenta and different boundary terms. Moreover, we are not sure if the naive application of the prescription found in the Wilson loop case is legal. Anyhow, the 3-form field in the M5-brane worldvolume is

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quite special. We have quite poor knowledge on it. Nevertheless, the action of the string soliton solutions did catch the essential properties of the multi-wound Wilson surface and the multi-Wilson surfaces. It would be very interesting to have a field theory calculation of the expectation values of the Wilson surfaces.

The surface operators could also be an important order parameter in four-dimensional gauge field theory. It has been used to study the geometric Langlands programme with ramification [43]. The bubbling geometry picture of the surface operators in $\mathcal{N} = 4$ SYM has been proposed in [44]. It would be interesting to illuminate the relations between the surface operators in four-dimensional SYM and the Wilson surface operators in six-dimensional (2,0)-theory.

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7. Various connections

In this appendix, we list various connections appeared in our calculation. For the induced metric (3.9), its Christoffel symbol has nonvanishing components:

$$
\begin{aligned}
\Gamma^t_{rt} &= \Gamma^x_{xr} = -\frac{f'}{f} \\
\Gamma^r_{tt} &= -\Gamma^r_{xx} = -\frac{f'}{(1+f'^2)f}, \\
\Gamma^r_{rr} &= -\frac{f'}{f} + f'\frac{f''}{1+f'^2} \\
\Gamma^r_{\alpha\alpha} &= \frac{1}{1+f'^2}\left(-r + \frac{f'}{f}r^2\right) \\
\Gamma^r_{\beta\beta} &= \frac{1}{1+f'^2}\left(-r + \frac{f'}{f}r^2\right)\sin^2\alpha \\
\Gamma^r_{\gamma\gamma} &= \frac{1}{1+f'^2}\left(-r + \frac{f'}{f}r^2\right)\sin^2\alpha\sin^2\beta \\
\Gamma^\alpha_{r\alpha} &= \Gamma^\beta_{r\beta} = \Gamma^\gamma_{r\gamma} = \left(\frac{1}{r} - \frac{f'}{f}\right) \\
\Gamma^\alpha_{\beta\beta} &= -\sin\alpha\cos\alpha, \\
\Gamma^\alpha_{\gamma\gamma} &= -\sin^2\beta\sin\alpha\cos\alpha \\
\Gamma^\beta_{\alpha\beta} &= \Gamma^\gamma_{\alpha\gamma} = \frac{\cos\alpha}{\sin\alpha}, \\
\Gamma^\beta_{\gamma\gamma} &= -\sin\beta\cos\beta \\
\Gamma^\gamma_{\beta\gamma} &= \frac{\cos\beta}{\sin\beta}
\end{aligned}
\tag{7.1}
$$

When $f = \kappa r$, some of the components above are vanishing.

For the AdS_7 spacetime, its nonvanishing independent components of spin connection are

$$\begin{aligned}
\omega^{\underline{1}}_{\underline{00}} &= -\frac{1}{R}, \quad \omega^{\underline{1}}_{\underline{ii}} = \frac{1}{R}, \quad \text{for } i \neq 0, 1 \\
\omega^{\underline{3}}_{\underline{ii}} &= -\frac{y}{Rr}, \quad \text{for } i = 4, 5, 6 \\
\omega^{\underline{4}}_{\underline{ii}} &= -\frac{y\cos\alpha}{Rr\sin\alpha}, \quad \text{for } i = 5, 6 \\
\omega^{\underline{5}}_{\underline{66}} &= -\frac{y\cos\beta}{Rr\sin\alpha\sin\beta}
\end{aligned} \tag{7.2}$$

For the metric (3.46), its Levi-Civita connection has components:

$$\begin{aligned}
\Gamma^{\alpha}_{\eta\alpha} &= \Gamma^{\beta}_{\eta\beta} = -\frac{1}{\sin\eta\cos\eta}, \qquad \Gamma^{\alpha}_{\beta\beta} = -\sin\alpha\cos\alpha, \quad \Gamma^{\beta}_{\alpha\beta} = \frac{\cos\alpha}{\sin\alpha} \\
\Gamma^{\eta}_{\eta\eta} &= -\frac{\cos\eta(1+2\kappa^2\sin^2\eta)}{\sin\eta(1+\kappa^2\sin^2\eta)}, \quad \Gamma^{\eta}_{\alpha\alpha} = \frac{(1+\kappa^2\sin^2\eta)\cos\eta}{(1+\kappa^2)\sin\eta}, \\
\Gamma^{\eta}_{\beta\beta} &= \frac{(1+\kappa^2\sin^2\eta)\cos\eta\sin^2\alpha}{(1+\kappa^2)\sin\eta}.
\end{aligned} \tag{7.3}$$

The independent nonvanishing components of the spin connections with respect to the vielbeins (3.50) are

$$\begin{aligned}
\omega^{\underline{1}}_{\underline{22}} = \omega^{\underline{1}}_{\underline{33}} &= \frac{1}{R\cos\eta}, \quad \omega^{\underline{2}}_{\underline{33}} = -\frac{\sin\eta\cos\alpha}{R\cos\eta\sin\alpha}, \\
\omega^{\underline{1}}_{\underline{44}} = \omega^{\underline{1}}_{\underline{55}} = \omega^{\underline{1}}_{\underline{66}} = \omega^{\underline{1}}_{\underline{77}} &= \frac{\cos\eta}{R}, \\
\omega^{\underline{4}}_{\underline{55}} = \omega^{\underline{4}}_{\underline{66}} = \omega^{\underline{4}}_{\underline{77}} &= -\frac{\cosh\rho\sin\eta}{R\sinh\rho}, \\
\omega^{\underline{5}}_{\underline{66}} = \omega^{\underline{5}}_{\underline{77}} &= -\frac{\cos\theta\sin\eta}{R\sin\theta\sinh\rho}, \quad \omega^{\underline{6}}_{\underline{77}} = -\frac{\cos\gamma\sin\eta}{R\sinh\rho\sin\theta\sin\gamma}
\end{aligned} \tag{7.4}$$

For the metric (4.2), its Levi-Civita connection has components:

$$\begin{aligned}
\Gamma^{2}_{33} &= -\sin\zeta_2\cos\zeta_2, \quad \Gamma^{2}_{44} = -\sin\zeta_2\cos\zeta_2\sin^2\zeta_3, \\
\Gamma^{3}_{44} &= -\sin\zeta_3\cos\zeta_3, \quad \Gamma^{3}_{23} = \Gamma^{4}_{24} = \frac{\cos\zeta_2}{\sin\zeta_2}, \quad \Gamma^{4}_{34} = \frac{\cos\zeta_3}{\sin\zeta_3}.
\end{aligned} \tag{7.5}$$

where the index i indicates ζ_i. And the nonvanishing components of the spin connection of S^4 with respect to vielbeins (4.7) are

$$\omega^{\underline{1}}_{\underline{22}} = \omega^{\underline{1}}_{\underline{33}} = \omega^{\underline{1}}_{\underline{44}} = -\frac{2\cos\zeta_1}{R\sin\zeta_1}, \tag{7.6}$$

$$\omega^{\underline{2}}_{\underline{33}} = \omega^{\underline{2}}_{\underline{44}} = -\frac{2\cos\zeta_2}{R\sin\zeta_1\sin\zeta_2}, \quad \omega^{\underline{3}}_{\underline{44}} = -\frac{2\cos\zeta_3}{R\sin\zeta_1\sin\zeta_2\sin\zeta_3} \tag{7.7}$$

For the metric (4.17), its nonvanishing Levi-Civita connection components are of the form

$$\begin{aligned}
\Gamma^{\delta}_{\delta\delta} &= -\Gamma^{\delta}_{\alpha\alpha} = \frac{\sin\delta}{\cos\delta}, \qquad \Gamma^{\delta}_{\beta\beta} = -\frac{\sin\delta}{\cos\delta}\sin^2\alpha \\
\Gamma^{\alpha}_{\delta\alpha} &= \Gamma^{\beta}_{\delta\beta} = \frac{1}{\sin\delta\cos\delta}, \quad \Gamma^{\alpha}_{\beta\beta} = -\sin\alpha\cos\alpha, \quad \Gamma^{\beta}_{\alpha\beta} = \frac{\cos\alpha}{\sin\alpha}
\end{aligned} \tag{7.8}$$

The spin connection with respect to the vielbein (4.22) has the components:

$$\omega_{\underline{ii}}^{\underline{1}} = \frac{1}{R}, \quad (\text{for } i = 2,3,4), \quad \omega_{\underline{33}}^{\underline{2}} = -\frac{y}{Rr_1},$$
$$\omega_{\underline{44}}^{\underline{2}} = -\frac{y}{Rr_1}, \qquad \omega_{\underline{44}}^{\underline{3}} = -\frac{y\cos\alpha}{Rr_1\sin\alpha}. \tag{7.9}$$

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